Universal renormalization-group dynamics at the onset of chaos in logistic maps and nonextensive statistical mechanics

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We uncover the dynamics at the chaos threshold μ_∞ of the logistic map and find it consists of trajectories made of intertwined power laws that reproduce the entire period-doubling cascade that occurs for $\mu < \mu_\infty$. We corroborate this structure analytically via the Feigenbaum renormalization group (RG) transformation and find that the sensitivity to initial conditions has precisely the form of a q-exponential, of which we determine the q-index and the q-generalized Lyapunov coefficient λ_q. Our results are an unequivocal validation of the applicability of the non-extensive generalization of Boltzmann-Gibbs (BG) statistical mechanics to critical points of nonlinear maps.

Critical points of nonlinear maps offer a suitable playground for testing the validity of the non-extensive generalization of the Botzmann-Gibbs (BG) statistical mechanics proposed by Tsallis over a decade ago [1,2]. Here we describe universal properties related to the dynamics of iterates at the onset of chaos in unimodal maps [3], that provide a literal confirmation of the generalized non-extensive theory. To this end we employ the celebrated one-dimensional logistic map, $f_\mu(x) = 1 - \mu |x|^2$, $-1 \leq x \leq 1$, and the properties of its renormalization group (RG) fixed point, to present evidence of previously unexposed scaling properties at the onset of chaos $\mu = \mu_\infty$. At this state, the most prominent of the map critical points, the trajectories of the iterates exhibit an intricate structure, that we describe and show is governed by the Feigenbaum's RG transformation [3].

The domain of validity of BG statistical mechanics has been implicitly challenged by the proposal of its non-extensive generalization. Subsequent studies have offered experimental and numerical evidence that point out both the inadequacy of the standard BG statistics and the plausible competence of the generalized theory in describing various types of phenomena and systems. This theoretical development represents an exceptional event in the long and trustworthy history of BG statistical mechanics. However, it is still in the process of being converted into a rigorously corroborated and fully understood fact. The suggested circumstances under which the generalized theory is believed to be applicable, at least with regards to non-linear dynamical systems, are those associated to a phase space with power-law sensitivity to initial conditions, to the consequent vanish-ing of the largest Lyapunov exponent, and to a fractal, or multifractal geometrical structure [2]. Here we show that our results for the dynamics at the onset of chaos in unimodal maps constitute an unequivocal proof of the universal validity of the non-extensive statistics at such critical points.

In fact, at the chaos threshold (as well as at other critical points of the map) the Lyapunov exponent λ_1 vanishes, and the sensitivity to initial conditions ξ_t, for large iteration time t, ceases to obey exponential behavior, exhibiting instead power-law behavior [4]. In order to describe the dynamics at such critical points, the q-exponential expression

$$\xi_t = \exp_q(\lambda_q t) \equiv [1 - (q-1)\lambda_q t]^{-1/(q-1)}, \qquad (1)$$

containing a q-generalized Lyapunov coefficient λ_q, has been proposed [5]. This expression is based on the non-extensive entropy of Tsallis [2]. As a companion to this, generalizations for the Kolmogorov-Sinai (KS) entropy K_q and for the Pesin identity $\lambda_q = K_q$, $\lambda_q > 0$ have also been introduced [5] (the standard expressions are recovered when $q \to 1$). Several recent studies [5–8], that probed numerically the onset of chaos of the logistic map and its generalization to nonlinearity $\zeta > 1$ ($f_{\mu,\zeta}(x) \equiv 1 - \mu |x|^\zeta$, $-1 \leq x \leq 1$), have revealed a series of precise connections between the Tsallis entropic index q and the map basic parameters. Here we present RG analytical results that corroborate the previously known value of q at μ_∞ for $\zeta = 2$ and also determine λ_q for the first time.

To state our results more precisely, we recollect the following properties. The logistic map exhibits several types

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of infinite sets of critical points that appear as its control parameter μ varies; these correspond, amongst others, to period-doubling and chaotic-band-splitting transitions [3]. The accumulation point of the period doublings and also of the band splittings is the Feigenbaum attractor that marks the threshold between periodic and chaotic orbits, at $\mu_\infty = 1.40115...$. The locations of period doublings (at $\mu = \mu_n < \mu_\infty$) and band splittings (at $\mu = \widehat{\mu}_n > \mu_\infty$) obey, for large n, power laws of the form $\mu_n - \mu_\infty \sim \delta^{-n}$ and $\mu_\infty - \widehat{\mu}_n \sim \delta^{-n}$, where $\delta = 4.6692...$ is one of the two Feigenbaum's universal constants. For our use below, we recall also the sequence of parameter values $\overline{\mu}_n$ employed to define the diameters of the bifurcation forks d_n that form the period-doubling cascade sequence. At $\mu = \overline{\mu}_n$ the map displays a 'superstable' periodic orbit of length 2^n that contains the point $x = 0$. For large n, the distances to $x = 0$ of the iterate positions in such 2^n-cycle that are closest to $x = 0$, $d_n \equiv f_{\overline{\mu}_n}^{(2^{n-1})}(0)$, have constant ratios: $d_n/d_{n+1} = -\alpha$, where $\alpha = 2.50290...$ is the second of the Feigenbaum's constants. A set of diameters with scaling properties similar to those of d_n can also be defined for the band splitting sequence [3]. For clarity of presentation of our results we shall only use absolute values of positions, so that the dynamics of iterates do not carry information on the self-similar properties of "left" and "right" symbolic dynamic sequences [3]. This choice does not affect results on the sensitivity to initial conditions. Below, d_n means $|d_n|$.

The main points in the following analysis are: 1) The iterates *at* μ_∞ follow trajectories that proceed in a concerted manner according to the entire period-doubling cascade that takes place for $\mu < \mu_\infty$. The positions of the trajectories are given in fact in terms of the diameters $d_n(\overline{\mu}_n)$ of the 2^n-supercycles. 2) As a consequence, the sensitivity to initial conditions also evolves in agreement to the period-doubling cascade. 3) The bounds, or envelops, as well as other monotonic subsequences, of both a single trajectory x_t and of the sensitivity to initial conditions ξ_t have precisely the form of a q-exponential. For ξ_t we have $q = 1 - \ln 2 / \ln \alpha$ and $\lambda_q = \ln \alpha / \ln 2$. 4) These results are obtainable via the fixed-point solution $g(x)$ of the RG doubling transformation, consisting of functional composition and rescaling: $\mathbf{R}f(x) \equiv \alpha f(f(x/\alpha))$.
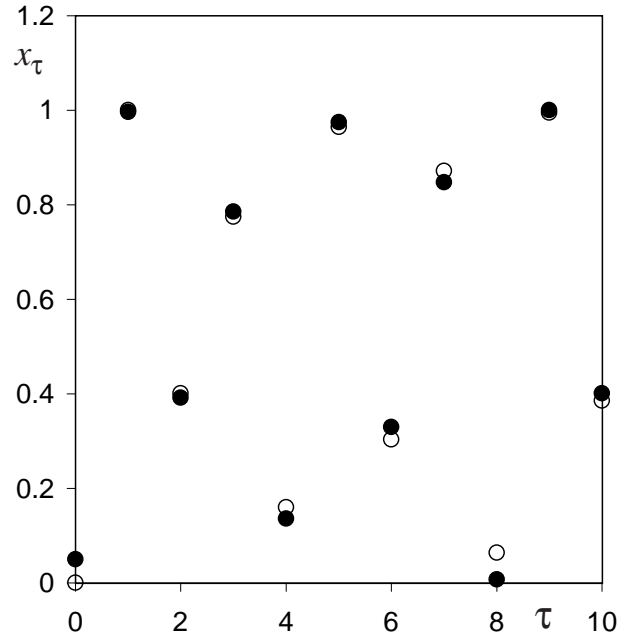

FIG. 1. Absolute values of positions of the first 10 iterations τ for two trajectories of the logistic map with initial conditions $x_0 = 0$ (empty circles) and $x_0 = \delta \simeq 5\ 10^{-2}$ (full circles). Plotted quantities are dimensionless.

To begin, we show in Fig. 1 the absolute values of the positions of two trajectories of the logistic map with initial conditions $x_0 = 0$ and $x_0 = \delta \simeq 5\ 10^{-2}$, for the first 10 iterations τ. For $\tau = 1$ the positions are $x_1 = 1$ and $x_1 \simeq 1 - 3.5\ 10^{-3}$, and it can be observed that the difference between the two positions at times $\tau = 2, 4$, and 8 grows progressively. In Fig. 2 we show the same first trajectory $x_0 = 0$ and a second one $x_0 = \delta \simeq 10^{-4}$, up to $\tau = 1000$. In the logarithmic scales it can be clearly appreciated that they consist of interwoven monotonic position subsequences with power-law decay. We are interested in the position subsequences that are generated by the time subsequences $\tau = 2^n + 2^{n-k}$, $k = 0, 1, ...$. As we show below, the values for the trajectory subsequence $\tau = 2^n$, with $x_0 = 0$, are asymptotically given by $x_\tau = d_n$, $n \geq 0$. More generally, each position subsequence $\tau = 2^n + 2^{n-k}$, obtained with $x_0 = 0$, is given by $g^{(2^k+1)}(0)\ d_n^k$, where $d_n^k \equiv f_{\overline{\mu}_n}^{(2^{n-k-1})}(0)$ and $n \geq k$. The itinerary of the iterate starting at $x_0 = 0$ can be clearly observed in Fig. 2. The position approaches the origin $x_0 = 0$ progressively as n increases every time that $\tau = 2^n$, but in between the values 2^n and 2^{n+1} it returns in an oscillatory manner towards $x_1 = 1$, repeating twice the positions visited in the previous cycle between 2^{n-1} and 2^n and introducing a new position between these two sub-cycles. For small n the positions are approximately repeated, but they become accurately reproduced as n increases. The whole time series has the period-doubling structure.
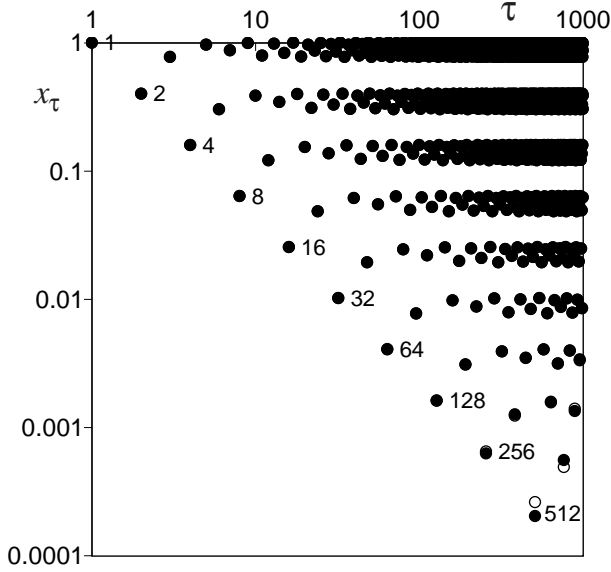
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FIG. 2. Absolute values of positions of the first 1000 iterations τ for two trajectories of the logistic map with initial conditions $x_0 = 0$ (empty circles) and $x_0 = \delta = 10^{-4}$ (full circles) in logarithmic scales. The power-law decay of several time subsequences can be clearly appreciated.

To interpret the dynamics in Figs. 1 and 2 in terms of the RG transformation, we consider $\mathbf{R}$ applied n times to the fixed-point map $g(x)$, i.e.:

$$g(x) = \mathbf{R}^{(n)}g(x) \equiv \alpha^n g^{(2^n)}(x/\alpha^n). \qquad (2)$$

We determine the trajectory positions at times $\tau = 2^n$, with $x_0 = 0$. Since $g(0) = 1$, we have $g^{(2^n)}(0) = \alpha^{-n}$ and because $d_n/d_{n+1} = \alpha$ with $d_0 = 1$ implies $d_n = \alpha^{-n}$, we also have $d_n = g^{(2^n)}(0)$. Thus, we obtain the diameters d_n to be the positions x_{2^n}. This result can be expressed as a q-exponential if we shift the time variable by one unit, $t = 2^n - 1$, and rearrange α^{-n} as $(1+t)^{-\ln\alpha/\ln 2}$. We obtain

$$x_t = \exp_Q(\Lambda_Q t), \qquad (3)$$

with $Q = 1 + \ln 2/\ln\alpha$ and $\Lambda_Q = -\ln\alpha/\ln 2$. Other position subsequences $\tau = 2^n + 2^{n-k}$ can be put in the form of q-exponentials with the same values of Q and Λ_Q.

The expression for the sensitivity to initial conditions can be derived with the use of the following approximate property (that becomes asymptotically exact in the limit $n \to \infty$)

$$g^{(2^n)}(d_j) = \frac{1}{\alpha^n} - \frac{\mu_\infty}{\alpha^{2j-n}} = d_n - \mu_\infty d_{2j-n}, \ n \le j. \quad (4)$$

To prove the above, note first that $g^{(2^n)}(d_j) = g^{(2^n+2^j)}(0) = \alpha^{-n}g^{(2^{j-n}+1)}(0)$, or, since $g^{(2^{j-n}+1)}(0) = g(d_{j-n})$, $g^{(2^n)}(d_j) = \alpha^{-n}g(d_{j-n})$. The preceding equality, together with $g(d_{j-n}) = 1 - \mu_\infty d_{j-n}^2 = 1 - \mu_\infty\alpha^{2n-2j}$, yields Eq. (4). Now, the distance between

positions $x_{2^n}(d_j) = g^{(2^n)}(d_j)$ and $x_{2^n}(d_i) = g^{(2^n)}(d_i)$ at time $\tau = 2^n$ can be written with the use of Eq. (4) as $x_{2^n}(d_j) - x_{2^n}(d_i) = [x_{2^0}(d_j) - x_{2^0}(d_i)]\alpha^n$, or with use of the shifted time variable $t = 2^n - 1$ as

$$x_t(d_j) - x_t(d_i) = (x_0(d_j) - x_0(d_i))\alpha^n. \qquad (5)$$

The sensitivity to initial conditions ξ_t is defined as

$$\xi_t \equiv \lim_{|\Delta x_0|\to 0} |x_t(d_j) - x_t(d_i)| \, / \, |x_0(d_j) - x_0(d_i)| \quad (6)$$

(where $\lim_{|\Delta x_0|\to 0}$ is equivalent to $\lim_{i,j\to\infty,i\ne j}$). ξ_t can then be written, considering that $\alpha^n = (1+t)^{\ln\alpha/\ln 2}$, as the q-exponential

$$\xi_t = \exp_q(\lambda_q t), \qquad (7)$$

where $q = 1 - \ln 2/\ln\alpha$ and $\lambda_q = \ln\alpha/\ln 2$. Notice that $q = 2 - Q$ as $\exp_q(y) = 1/\exp_Q(-y)$. The previous construction applies strictly to initial positions that lie on the attractor; nevertheless, we remark that all other positions tend to the attractor with a power-law behavior (see [10,11]), so that after an initial transient their dynamics becomes practically indistinguishable from the situation we describe here. In Fig. 3 we show the q-logarithm of ξ_t vs t (with $q = 1 - \ln 2/\ln\alpha = 0.2445...$), from a numerical simulation of two trajectories with initial conditions $x_0 = 0$ and $x_0 = \delta \simeq 10^{-8}$. The result is a straight line with slope very close to $\lambda_q = \ln\alpha/\ln 2 = 1.3236...$. This corroborates the RG prediction (the q-logarithm, $\ln_q y \equiv (y^{1-q} - 1)/(1-q)$, is the inverse of $\exp_q(y)$).
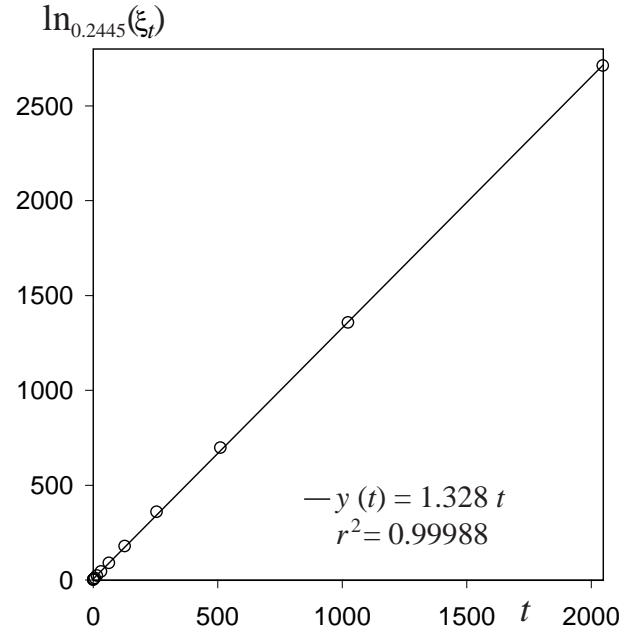

FIG. 3. The q-logarithm of sensitivity to initial conditions ξ_t vs t, with $q = 1 - \ln 2/\ln\alpha = 0.2445...$ and initial conditions $x_0 = 0$ and $x_0 = \delta \simeq 10^{-8}$ (circles). The full line is the linear regression $y(t)$. As required, the numerical results reproduce a straight line with a slope very close to $\lambda_q = \ln\alpha/\ln 2 = 1.3236...$.

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Interestingly, both x_t/x_0 and ξ_t can be seen to satisfy the dynamical fixed-point relations $h(t) = \alpha h(h(t/\alpha))$ with $\alpha = 2^{1/(Q-1)}$ and $\alpha = 2^{1/(q-1)}$, respectively. In relation to this, we note that the static fixed-point solution $f^*(x)/x$ to the Feigenbaum RG recursion relation for the case of the tangent bifurcation, obtained by Hu and Rudnick [3,9] to study the intermittency transition in the ζ-logistic map, has the form of a q-exponential with $q = 2$. This has been pointed out recently [10,11], where in addition it has been shown that this solution applies too, but now with $q = 3$, to the period-doubling transitions that take place at $\mu_n < \mu_\infty$. We recall that, for the transition to periodicity of order n, the RG transformation is applied to the n-th composition $f^{(n)}$ of the original map in the neighborhood of one of the n points tangent to the line with unit slope, and a shift is made of the origin of coordinates to that point. The RG fixed-point map is $f^*(x) = x \exp_q(u x^{q-1})$, where u is the leading expansion coefficient of $f^{(n)}$ and where the recursion relation is satisfied with $\alpha = 2^{1/(q-1)}$. In the neighborhood of the intermittency and period-doubling transitions the time evolution of iterates follow monotonic paths set by the form of the map itself, and are also q-exponentials. For these types of critical points the static and dynamic properties are simply related and obey expressions of the same form [10,11]. Not only the entropic index q can be plainly identified, but the q-generalized Lyapunov coefficient λ_q turns out to be given by the expansion coefficient u. Unequivocal corroboration of these results has been obtained recently [11].

At the onset of chaos both static and dynamical properties are more complex. The celebrated RG fixed-point map static solution is obtained as a power series of a smooth unimodal trascendental function, the universal $n \to \infty$ limit of $(-\alpha)^n f_{\overline{\mu}_{n+1}}^{(2^n)}(x/(-\alpha)^n)$; likewise, the fractal dimension of the attractor $d_f = 0.538045143...$ is obtained considering also the same $n \to \infty$ limit on positions of the 2^n-cycles [3,12]. As we have seen here, the multifractal attractor at μ_∞ imprints an involved structure into the time evolution of the iterates, that can be resolved in terms of simpler monotonic time subsequences. Remarkably, these subsequences and the sensitivity to initial conditions ξ_t are analytically reproduced by the same RG transformation originally applied to describe static properties. These quantities evolve as universal q-exponentials, with q and λ_q simply expressed in terms of α. We observe then a connection between dynamic properties of a strange attractor at the edge of chaos, such as the q-generalized Lyapunov coefficient, and the static properties, described by the set of distances d_n^k that make up this multifractal; this is in the spririt of the Kaplan-Yorke conjecture [3].

Under some conditions, exemplified here by critical points in nonlinear maps, the Lyapunov exponents of a system that measure the strength of phase space mixing vanish. When this happens dynamic processes become sluggish in exploring their permissible configurations and may be capable of covering only a small fraction of the available phase space, even in the limit $t \to \infty$. This fraction may have a fractal dimension smaller than the total dimension of phase space. These are thought to be the conditions for failure of BG statistics and applicability of its nonextensive generalization [2], and the significance of our analytically-backed results with no approximations is a contribution towards the clarification of this issue. At the onset of chaos of unimodal maps the reduced phase subspace is represented by the strange attractor, a Cantor subset of the interval $-1 \le x \le 1$. It is important to point out that in this case the permissible positions (configurations) are asymptotically confined by the attractor and this acts as an inescapable barrier to movement to other locations. By construction, the dynamics at the onset of chaos, as well as those restricted to the neighborhood of the intermittency transitions, describe a purely nonextensive regime. It should be mentioned that our analysis does not consider the access of trajectories to an adjacent or neighboring chaotic region, as in the setting of Refs. [13,14] or that in conservative maps [15]. Hence there is no feedback vehicle for a crossover from a nonextensive regime with vanishing ordinary Lyapunov exponent to an extensive regime with a positive one at some t_{cross} as $t \to \infty$.

Clearly, our findings have wide-ranging validity, as they apply to all dissipative systems in the Feigenbaum universality class. It might be possible to observe analogous behavior at chaos boundary criticalities in other classes of dynamical systems, such as period doubling in bimodal [16] and other multiparameter maps [17], and quasiperiodicity in dissipative systems [18]. More generally, the properties discussed here are likely to hold strictly for other types of systems or situations that possess equivalent phase space limitations. Those systems for which experimental and numerical evidence has accumulated on BG statistics inadequacy and nonextensive statistics competency [2] warrant examination. The more that is learned on mechanisms and circumstances leading to a hindered phase space the clearer the physical understanding of the applicability of the nonextensive theory will become.

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